Midsemestral exam
Second semester 2009-10
M.Math. Ist year
Answer any 6 - Be brief !

Q 1.
Suppose that a plane curve α of unit speed is such that its tangent at any point makes the same angle θ with $\alpha(s)$ for any s. If $0 < \theta < \pi/2$, prove that the curve must be a logarithmic spiral $\alpha(t) = (e^{ct}\cos(ct), e^{ct}\sin(ct))$ for a constant c. Also, determine the constant c in terms of θ.
Hint : Find the signed curvature.

Q 2.
Show that Pascal's limacon $\alpha(t) = ((1 + 2\cos t)\cos t, (1 + 2\cos t)\sin t)$ has only two vertices. Why does it not contradict the four vertex theorem?

Q 3.
If the 3rd derivative $\alpha^{(3)}(s) \neq 0$, show that the points $\alpha(s + h)$ for small $h > 0$ are on the same side of the osculating plane as $\alpha^{(3)}(s)$ and the points $\alpha(s - h)$ for small $h > 0$ are on the other side.

Q 4.
Let $\alpha : (a, b) \to \mathbf{R}^2$ is a regular curve and let $t_0 \in (a, b)$ such that $|\alpha(t_0)| = max(|\alpha(t)| : t \in (a, b))$. Prove that $|K(t_0)| \geq \frac{1}{|\alpha(t_0)|}$.

Q 5.
Consider the parametrized curve $\alpha : \mathbf{R} \to \mathbf{R}^3$ given by :
$t \mapsto (t, 0, e^{-1/t^2})$ or $(t, e^{-1/t^2}, 0)$ or $(0, 0, 0)$ according as to whether
$t > 0, t < 0$ or $t = 0$.
Show that α is a differentiable regular curve whose curvature $K(t)$ is non-zero at all $t \neq 0, \pm\sqrt{2/3}$.

Q 6.
Compute the unit tangent, signed curvature, signed normal and torsion for the curve :
$\alpha(t) = (\frac{\cos t}{\sin t/2}, 2\cos t/2, \frac{\sqrt{3}}{\sin t/2})$.

OR

Compute the curvature at the point $(0, 0, 0)$ of the curve given by the implicit equations :

$$x + \sin hx = \sin y + y, z + e^z = x + \log(1 + x) + 1.$$

Q 7.
Show that a plane curve and its osculating circle at a point have a contact of order three or more. Further, the contact is of order four or more if and only if $K'_{sign} = 0$ at the contact point.

OR

Show that a curve in $\mathbf{R}^3$ and its osculating plane at a point have a contact of order three or more, and of order four or more if and only if the torsion $t = 0$ at the contact point.

Q 8.
If the tangents to a curve are parallel to a certain plane, prove that the curve is planar.

Q 9.
For a space curve given by the parametrized equation $r = r(t)$ not necessarily in terms of arc length, assuming that the curvature does not vanish anywhere, show that the curvature $K(t)$ and torsion $\tau(t)$ are given, respectively, by :

$$||r' \times r''||/||r'||^3$$

$$(r' \times r'').r''')||/||r' \times r''||^2.$$